tinct, *coincident*, *or imaginary*, *according as the line intersects the base conic in two real*, *coincident*, *or imaginary points.*

The conic which transforms into $y=mx+l$, given by substituting from (3) into $y_1=mx_1+l$, is $bdx^2(l+mc)+acy^2(l-b)+bxy(mac-cd+ldh)$
$+abdx(l+mc)+acdy(l-b)=0\ldots.(10)$.

This intersects the conic (6) into which $y=mx+l$ transforms in four points, which, if we subtract (6) from (10) and factor, are seen to lie on

$$\{acy(b+d)+bdx(a+c)\}\{mx-y+l\}=0.$$

O is one of the two intersections lying on OK. Call the other H. Then the point in which $y=mx+l$ meets OK and the point H mutually correspond. *We thus have an involution marked out on* OK.

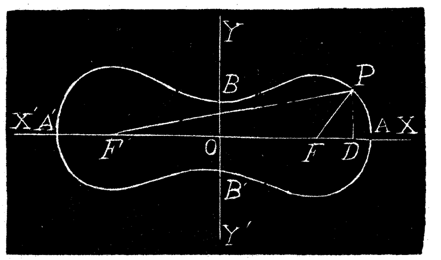
We saw above that the points A, D, O transform into the lines AC, DB, BC respectively. Now we can prove either geometrically or analytically that the lines AD, AO, DO transform into the points O, C, B respectively. Thus the sides and vertices of $\triangle ADO$ transform into the vertices and sides of $\triangle OBC$. With this exception the correspondence between the points in the two systems is one to one. The *projective* treatment of this transformation and its dual will be given elsewhere.

[For Projective Treatment, see my paper in the *Rendiconti del Circolo di Palermo.*]

THE RECTIFICATION OF THE CASSINIAN OVAL BY MEANS OF ELLIPTIC FUNCTIONS.

By F. P. MATZ, Sc. D., Ph. D., Mechanicsburg, Pennsylvania.

The Cassinian Oval is the locus of a point the *product* of whose distances from two fixed points is constant.



Let P be any point on the curve, F and F' the foci, O the middle point of FF', $OD=x$, $DP=y$, $OF=c$, $FP=\rho$, and $F'P=\rho'$; then, according to the definition of the curve, $\rho\rho'=m^2\ldots.(1)$. From the diagram, $\rho=\pm\sqrt{[(x-c)^2+y^2]}$ and $\rho'=\pm\sqrt{[(x+c)^2+y^2]}$; that is, from (1) we obtain the equation.

$$\sqrt{\{[(x-c)^2+y^2]\times[(x+c)^2+y^2]\}}=m^2\ldots.(a).$$

$\therefore (x^2+y^2+c^2)^2-4c^2x^2=m^4\ldots.(2)$; and this is the Cartesian equation of the Cassinian Oval, the co-ordinate axes being rectangular.

Put $OP=r$, and the $\angle POD=(\frac{1}{2}\pi-\theta)$; then $OD=x=r\sin\theta$, and $PD=y=r\cos\theta$. From (2), therefore, we have $r^4+2c^2(1-2\sin^2\theta)r^2=m^4-c^4\ldots.(b)$, or $r^4+(2c^2\cos 2\theta)r^2=m^4-c^4\ldots.(3)$, which is a convenient

form for the *central-polar* equation of the Cassinian Oval.

I. RECTIFICATION.—In order to *rectify* the Cassinian Oval, we deduce, from (3)

$$\cos 2\theta = \frac{(m^4-c^4)-r^4}{2c^2r^2} \ldots (c). \quad \therefore \sin 2\theta = \sqrt{\left(\frac{4c^4r^4-[(m^4-c^4)-r^4]^2}{4c^4r^4}\right)} \ldots (d).$$

$$\therefore \frac{d\theta}{dr} = \frac{(m^4-c^4)-r^4}{2c^2r^3 \sin 2\theta}, \text{ and } \left(\frac{rd\theta}{dr}\right)^2 = \frac{[(m^4-c^4)+r^4]^2}{4c^4r^4-[(m^4-c^4)-r^4]^2} \ldots (e).$$

[The following transformation of (c) affords a *second* method for the derivation of (e):

$$\theta = \tfrac{1}{2}\cos^{-1}\left(\frac{(m^4-c^4)-r^4}{2c^2r^2}\right) = \cos^{-1}\left[\sqrt{\left(\frac{(m^4-c^4)-r^4}{4c^2r^2}+\tfrac{1}{2}\right)}\right]$$

$$= \cos^{-1}\left(\frac{\sqrt{[m^4-(c^2-r^2)^2]}}{2cr}\right), = \sin^{-1}\left(\frac{\sqrt{[(c^2+r^2)^2-m^4]}}{2cr}\right) \ldots (f).]$$

From (c), when $\theta=0$, we have $r=\pm\sqrt{(m^2-c^2)}$, $=\pm b$; also, when $\theta=\frac{1}{2}\pi$, we have $r=\pm\sqrt{(m^2+c^2)}$, $=\pm a$. Since the perimeter of the Cassinian Oval is composed of four equal quadrantal arcs,

$$\mathsf{P} = 4\int_b^a \sqrt{\left(\frac{4c^4r^4-[(m^4-c^4)-r^4]^2+[(m^4-c^4)+r^4]^2}{4c^4r^4-[(m^4-c^4)-r^4]^2}\right)}dr$$

$$= 8m^2\int_b^a \frac{r^2dr}{\sqrt{\{[(c^2+r^2)^2-m^4]\times[m^4-(c^2-r^2)^2]\}}} = 8m^2.$$

$$\int_b^a \frac{r^2dr}{\sqrt{\{[(m^2+c^2)+r^2]\times[(m^2+c^2)-r^2]\times[r^2+(m^2-c^2)]\times[r^2-(m^2-c^2)]\}}}$$

$$= 8m^2\int_b^a \frac{r^2dr}{\sqrt{\{[(m^2+c^2)^2-r^4]\times[r^4-(m^2-c^2)^2]\}}} \ldots (4).$$

Put $\quad r^4=(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi \ldots (g); \quad$ then
$4r^3dr=2[-(m^2+c^2)^2+(m^2-c^2)^2]\sin\phi\cos\phi d\phi \ldots (h).$

$$\therefore r^2dr = \frac{-2m^2c^2\sin\phi\cos\phi d\phi}{[(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi]^{\frac{1}{4}}} \ldots (i).$$

Transforming (4) by means of (g) and (i), the expression for the required perimeter becomes

$$\mathsf{P} = 8m^2\int_0^{\frac{1}{2}\pi} \frac{2m^2c^2[(m^2+c^2)\cos^2\phi+(m^2-c^2)^2\sin^2\phi]^{-\frac{1}{4}}\sin\phi\cos\phi d\phi}{\sqrt{\{[(m^2+c^2)^2-(m^2-c^2)^2]\sin^2\phi\times[(m^2+c^2)^2-(m^2-c^2)^2]\cos^2\phi\}}}$$

$$= 4m^2\int_0^{\frac{1}{2}\pi} \frac{d\phi}{[(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi]^{\frac{1}{4}}}$$

$$= 4m^2\int_0^{\frac{1}{2}\pi} \frac{d\phi}{\{(m^2+c^2)^2-[(m^2+c^2)^2-(m^2-c^2)^2]\sin^2\phi\}^{\frac{1}{4}}}$$

$$= 4m^2\int_0^{\frac{1}{2}\pi} \frac{d\phi}{[(m^2+c^2)^2-4m^2c^2\sin^2\phi]^{\frac{1}{4}}} = \frac{4m^2}{\sqrt{(m^2+c^2)}}\int_0^{\frac{1}{2}\pi} \frac{d\phi}{[1-C^2\sin^2\phi]^{\frac{1}{4}}}$$

$$= \frac{4m^2}{\sqrt{(m^2+c^2)}} \int_0^{\frac{1}{2}\pi} \left[1 + \frac{C^2 \sin^2\phi}{4} + \frac{5C^4 \sin^4\phi}{32} + \frac{15C^6 \sin^6\phi}{128} + \text{etc.} \right] d\phi$$

$$= 2\pi \sqrt{\left(\frac{m^4}{m^2+c^2}\right)} \left[1 + \frac{C^2}{8} + \frac{15C^4}{256} + \frac{75C^6}{2048} + \text{etc.} \right] \ldots (5), \text{ in which}$$

$C^2 = 4m^2c^2 / (m^2+c^2)^2$. When $c=2$ and $m^4=25$, we deduce from (5) the numerical result, $\mathsf{P}=12.7329+$, expressing the required perimeter. With respect to the *locus* represented by (3), we have the following hypotheses: $m>c \ldots (\alpha)$, $m=c \ldots (\beta)$, and $m<c \ldots (\gamma)$. Under the first hypothesis, the said locus represents the *Cassinian Oval*; under the second hypothesis, the said locus represents the *Bernoullian Lemniscate*; and under the third hypothesis, the said locus represents two *ovaliform figures*.

In rectifying under the hypotheses (β) and (γ), the term $(m^2-c^2)^2$ in (4) and in (g) must be altered accordingly.

[To be continued.]

THE CONSTRUCTION OF THE SUN'S PATH.

By ERIC DOOLITTLE, Professor of Mathematics in the State University of Iowa, Iowa City, Iowa.

In the *Archiv der Mathematik and Physik*, Vol. LIII. Part IV., there is a very interesting article by Professor Hoza on the graphical construction of the sun's apparent path. He considers the earth as stationary in its orbit during a perod of twenty-four hours, and obtains the projection of the apparent path during that time upon the plane of the Equator; the result being, as might be expected, an ellipse, the ratio of whose axes is as 1 : cos δ.

This construction only applies to those places on the earth where the sun actually rises and sets each day, nor is the exact path thus found, since the sun's motion is not taken into account. An investigation of this latter is not difficult; it will lead us to a very interesting spiral curve.

Let us take the vernal equinox as an origin; the arc of the equator as an axis of X positive toward the right, and a great circle perpendicular to the equator through the vernal equinox as the axis of Y. The circles lie on the celestial sphere, whose radius is considered as unity.

Then, if c be the angular velocity of the earth on its axis, and K that of the sun in the ecliptic, (considered as uniform), we will have at a time t after the time of vernal equinox: